Equations of Mathematical Physics and Compositions of Brownian and Cauchy processes

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Abstract

We consider different types of processes obtained by composing Brownian motion $B(t)$, fractional Brownian motion $B_H(t)$ and Cauchy processes $C(t)$ in different manners.

We study also multidimensional iterated processes in $\mathbb{R}^d$, like, for example, $(B_1(|C(t)|),...,B_d(|C(t)|))$ and $(C_1(|C(t)|),...,C_d(|C(t)|))$, deriving the corresponding partial differential equations satisfied by their joint distribution.

We show that many important partial differential equations, like wave equation, equation of vibration of rods, higher-order heat equation, are satisfied by the laws of the iterated processes considered in the work.

Similarly we prove that some processes like $C(|B_1(|B_2(...|B_{n+1}(t)|...)|)|)$ are governed by fractional diffusion equations.

Key words: Iterated Brownian motion; Fractional partial differential equations; Riesz fractional derivative; Fractional Brownian motion; Cauchy process; Wave equation; Vibration of rods.

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1 Introduction

The iterated Brownian motion is obtained by the composition of two independent Brownian motions B_1 and B_2, as follows:

$$I(t) = B_1(|B_2(t)|), \qquad t > 0. \tag{1.1}$$

Recently this kind of processes has been studied by many authors: see, for example, Burdzy (1994), Khoshnevisan and Lewis (1996), Allouba and Zheng (2001), Allouba (2002), Beghin and Orsingher (2009). As far as the applications are concerned, it has been observed that the iterated Brownian motion I is suitable to describe diffusions in cracks (De Blassie (2004)) and many other physical phenomena. In particular, fractures on a rectangular slab can be viewed at as trajectories of a Brownian motion (see Chudnovsky and Kunin (1987)). The flow of a gas on a fracture can be

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represented by a Brownian motion moving on a Brownian sample path and therefore this model produces an iterated Brownian motion (see, for example, Khoshnevisan and Lewis (1999)).

In Orsingher and Zhao (1999) it is stated that the density of the process

$$p_I(x,t) = 2\int_0^\infty \frac{e^{-\frac{x^2}{2s}}}{\sqrt{2\pi s}} \frac{e^{-\frac{s^2}{2t}}}{\sqrt{2\pi t}} ds \tag{1.2}$$

satisfies the fourth-order equation

$$\frac{\partial p}{\partial t}(x,t) = \frac{1}{2^3}\frac{\partial^4 p}{\partial x^4} + \frac{1}{2\sqrt{2\pi t}}\frac{d^2}{dx^2}\delta(x), \tag{1.3}$$

for $x \in \mathbb{R}$ and $t > 0$.

In De Blassie (2004) a similar result is proved under the initial condition $p(x,0) = f(x)$.

Moreover it has been shown in Orsingher and Beghin (2004) that the density (1.2) is a solution to the fractional equation

$$\frac{\partial^{\frac{1}{2}} p}{\partial t^{\frac{1}{2}}} = \frac{1}{2^{3/2}}\frac{\partial^2 p}{\partial x^2}, \qquad x \in \mathbb{R},\ t > 0 \tag{1.4}$$

(with initial condition $p(x,0) = \delta(x)$), which is a limiting case of the fractional telegraph equation

$$\frac{\partial p}{\partial t} + 2^{3/2}\lambda \frac{\partial^{\frac{1}{2}} p}{\partial t^{\frac{1}{2}}} = c^2 \frac{\partial^2 p}{\partial x^2}, \qquad x \in \mathbb{R},\ c, \lambda, t > 0,$$

for $\lambda, c \to \infty$, in such a way that $c^2/\lambda \to 1$.

Generalizations of this result have been obtained in many directions. First of all by considering the n-times iterated Brownian motion

$$I_n(t) = B_1(|B_2(...|B_{n+1}(t)|...)|), \tag{1.5}$$

involving $n+1$ independent one-dimensional Brownian motions.

It has been shown in Orsingher and Beghin (2009) that the law of (1.5) satisfies the following fractional equation

$$\frac{\partial^{\frac{1}{2^n}} p}{\partial t^{\frac{1}{2^n}}} = 2^{\frac{1}{2^n}-2}\frac{\partial^2 p}{\partial x^2}, \qquad x \in \mathbb{R},\ t > 0, \tag{1.6}$$

with $p(x,0) = \delta(x)$.

For the vector process

$$I^d(t) = \begin{cases} B_1(|B(t)|) \\ \cdots \\ \cdots \\ B_d(|B(t)|) \end{cases}, \qquad t > 0, \tag{1.7}$$

where B and $B_1, ..., B_d$ are mutually independent Brownian motions, it is proved in Orsingher and Beghin (2009) that the joint law $p_I^d = p_I^d(x_1, x_2, ..., x_d, t)$ is a solution to the fractional equation

$$\frac{\partial^{\frac{1}{2}} p}{\partial t^{\frac{1}{2}}} = \frac{1}{2^{3/2}}\sum_{j=1}^d \frac{\partial^2 p}{\partial x_j^2}, \qquad x_j \in \mathbb{R},\ j = 1, ..., d,\ t > 0. \tag{1.8}$$

The constants appearing in equations (1.4), (1.6) and (1.8) have been chosen in such a way that the Brownian motions involved in the construction of the iterated processes possess standard distribution.

If we consider the slightly more general fractional equation

$$\frac{\partial^{\frac{1}{2}} p}{\partial t^{\frac{1}{2}}} = \lambda^2 \sum_{j=1}^{d} \frac{\partial^2 p}{\partial x_j^2}, \qquad x_j \in \mathbb{R},\ j = 1, ..., d,\ t > 0, \tag{1.9}$$

then the joint density of the vector process reads

$$p_I^d(x_1, x_2, ..., x_d, t) = 2 \int_0^\infty \frac{e^{-\frac{w^2}{2(2^3\lambda^4 t)}}}{\sqrt{2\pi(2^3\lambda^4 t)}} \prod_{k=1}^{d} \frac{e^{-\frac{x_k^2}{2w}}}{\sqrt{2\pi w}} dw. \tag{1.10}$$

Result (1.10) shows that the components of the d-dimensional vector process $(B_1(|B(t)|), ..., B_d(|B(t)|))$ are no longer independent and the parameter λ enters into the variance of the "time process" $B(t), t > 0$.

For the law of the vector process $I^d(t)$ we can also write the equation which is analogous to (1.3):

$$\frac{\partial p}{\partial t}(x, t) = \frac{1}{2^3} \left(\sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2} \right)^2 p + \frac{1}{2\sqrt{2\pi t}} \sum_{j=1}^{d} \frac{d^2}{dx_j^2} \prod_{k=1}^{d} \delta(x_k), \tag{1.11}$$

with initial condition $p(x_1, ..., x_d, 0) = \prod_{k=1}^{d} \delta(x_k)$.

As far as the d-dimensional case is concerned, along with the definition (1.7) of the process I^d with components being independent one-dimensional iterated Brownian motions, another choice is to define the process as $B_{(d)}(|B(t)|), t > 0$, with $B_{(d)}$ being a d-dimensional Brownian motion independent of the Brownian motion $B(t)$. For the process $I^d(t)$ Allouba and Zheng (2001) and De Blassie (2004) show that the function $u(t, x) = E_x(f(I^d(t))) = E(f(I^d(t))|I_0^d = x)$ solves the initial value problem

$$\frac{\partial}{\partial t} u(t, x) = \Delta^2 u(t, x) + \frac{1}{\sqrt{\pi t}} \Delta f(x), \quad u(0, x) = f(x), \tag{1.12}$$

for $t > 0$, $x \in \mathbb{R}^d$, and 'sufficiently good' functions $f(x)$, $x \in \mathbb{R}^d$ (which are supposed to be bounded with bounded Hlder continuous second derivatives).

Moreover, when replacing the outer process $B_{(d)}$ with a continuous Markov process X and considering $Z(t) = X(|B(t)|)$, equation (1.12) remains valid for $u(t, x) = E_x(f(Z(t)))$ if we replace the Laplacian with the generator L_x of the continuous semigroup associated with the Markov process X, that is:

$$\frac{\partial}{\partial t} u(t, x) = L_x^2 u(t, x) + \frac{1}{\sqrt{\pi t}} L_x f(x), \quad u(0, x) = f(x),$$

$f \in D(L_x)$, where $D(L_x)$ is the domain of the operator L_x. Moreover $u(t, x) = E_x(f(Z(t)))$ solves the fractional Cauchy problem

$$\frac{\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}} u(t, x) = L_x u(t, x), \quad u(0, x) = f(x), \quad x \in \mathbb{R}^d,\ t > 0$$

(see, Baeumer, Meerschaert and Nane (2009)).

A further extension is given in Baeumer, Meerschaert and Nane (2009) (see also Nane (2008)). Let $X(t) = x + X_0(t)$, where $X_0(t)$ is a Lvy process in $\mathbb{R}^d$ starting at zero, and let L_x be the generator of the semigroup $p(x,t) = E_x(f(X(t)))$. Then for any $f \in D(L_x)$ and any $n = 2, 3, 4, ...$ it was shown that both Cauchy problems for the higher-order partial differential equation

$$\frac{\partial u}{\partial t} = \sum_{i=1}^{n-1} \frac{t^{\frac{j}{n}-1}}{\Gamma\left(\frac{j}{n}\right)} L_x^j f + L_x^n u, \quad u(x,0) = f(x), \qquad x \in \mathbb{R}^d, t > 0 \tag{1.13}$$

and for the fractional equation

$$\frac{\partial^{\frac{1}{n}} u}{\partial t^{\frac{1}{n}}} = L_x u, \quad u(x,0) = f(x), \quad x \in \mathbb{R}^d,\ t > 0, \tag{1.14}$$

have the same unique solution. This is given by $u(x,t) = E_x(f(X(E^{1/n}(t))))$, where $E^{1/n}(t)$ is the inverse Lvy subordinator of order $\frac{1}{n}$, that is, $E^{1/n}(t) = \inf\{s : D(s) > t\}$, where $D(s)$ is a stable subordinator with index $\frac{1}{n}$.

For the Markov process with the inverse Lvy subordinator of order $\frac{1}{2}$ one gets the same governing equations as in the case of a Markov process with a Brownian time, that is, these two processes have the same one-dimensional distributions.

In the light of the above equivalence of (1.13) and (1.14) (and inspired by the result of Orsingher and Beghin (2009), equation (1.6)), the following PDE connection of n-iterated Brownian motion appears in Nane (2008): the unique solution of the Cauchy problems (1.13) and (1.14) with $n = 2^k$ is given by $u(x,t) = E_x(f(X(I_{k-1}(t))))$, where X is the Lvy process described above and $I_k(t)$ is k-times iterated Brownian motion (1.5).

For the law of the n-times iterated Brownian motion, we can write an analog of equations (1.3) and (1.11):

$$\frac{\partial p}{\partial t} = \sum_{i=1}^{m-1} \frac{t^{\frac{j}{m}-1}}{\Gamma\left(\frac{j}{m}\right)} \left(\frac{\partial^2}{\partial x^2}\right)^j \delta(x) + \left(\frac{\partial^2}{\partial x^2}\right)^m p, \quad p(x,0) = \delta(x), \quad x \in \mathbb{R}, \quad t > 0, \tag{1.15}$$

where $m = 2^n$.

For the fractional diffusion equation of order $1/n$, $n \in \mathbb{N}$,

$$\frac{\partial^{\frac{1}{n}} p}{\partial t^{\frac{1}{n}}} = \frac{1}{2} \frac{\partial^2 p}{\partial x^2}, \qquad x \in \mathbb{R},\ t > 0, \tag{1.16}$$

with initial condition $p_{1/n}(x,t) = \delta(x)$, it has been shown in Beghin and Orsingher (2003) that the solution coincides with the density of the process

$$I_{1/n}(t) = B\left(\prod_{j=1}^{n-1} G_j(t)\right), \qquad n > 1, \tag{1.17}$$

where $(G_1(t), ..., G_n(t))$ is a random variable with joint distribution

$$p(w_1, ..., w_{n-1}) = \frac{n^{\frac{n-1}{2}}}{(2\pi)^{\frac{n-1}{2}} \sqrt{t}} e^{-\frac{w_1^n + \ldots + w_{n-1}^n}{\sqrt[n-1]{n^n t}}} w_2 ... w_{n-1}^{n-2}, \qquad w_j \geq 0,\ 1 \leq j \leq n-1. \tag{1.18}$$

Taking into account the result of Baeumer, Meerschaert and Nane (2009) on the solution to the equation (1.14), we can conclude that the process (1.17) and the process $B\left(E^{1/n}(t)\right)$, that is a Brownian motion subordinated to the inverse Lvy process of order $\frac{1}{n}$, have the same one-dimensional distributions, which can be written down in a closed form.

In D'Ovidio and Orsingher (2009) various types of processes obtained by composing independent fractional Brownian motions $B_{H_1}(t)$ and $B_{H_2}(t), t>0$ (with Hurst parameters respectively equal to $H_1, H_2 \in (0,1)$) have been studied. In particular, for the density $p_{H_1H_2}(x,t)$ of $B_{H_1}(|B_{H_2}(t)|)$ it has been shown that the governing equation has the following structure:

$$(1+H_1H_2)t\frac{\partial p}{\partial t}+t^2\frac{\partial^2 p}{\partial t^2}=H_1^2H_2^2\left\{2x\frac{\partial p}{\partial x}+x^2\frac{\partial^2 p}{\partial x^2}\right\}, \quad x\in\mathbb{R}, t>0.$$

In this paper we extend some of these results in several directions, by considering different compositions of Brownian motions, fractional Brownian motions and Cauchy processes.

We start by studying the iterated Brownian motion defined in (1.1) in the case where B_1 is endowed by drift, giving the two governing equations, the fractional and fourth-order one (see (2.4) below).

Another direction of our research concerns processes obtained by combining Brownian motions (possibly fracional Brownian motions) and Cauchy processes. In particular we study the multidimensional vector processes $(B_1(|C(t)|),...,B_d(|C(t)|))$ and $(C_1(|B(t)|),...,C_d(|B(t)|))$, which involve independent Brownian motions $B_j(t), t>0$ and Cauchy processes $C_j(t), t>0$.

For $d=1$ it is has been proved in D'Ovidio and Orsingher (2009) that the law of $B(|C(t)|), t>0$, given by

$$p_{BC}(x,t)=\frac{2}{\pi}\int_0^\infty \frac{e^{-\frac{x^2}{2s}}}{\sqrt{2\pi s}}\frac{t}{t^2+s^2}ds, \quad x\in\mathbb{R},\ t>0 \tag{1.19}$$

satisfies the following fourth-order equation

$$\frac{\partial^2 p}{\partial t^2}=-\frac{1}{2^2}\frac{\partial^4 p}{\partial x^4}-\frac{1}{\pi t}\frac{d^2}{dx^2}\delta(x), \qquad x\in\mathbb{R},\ t>0, \tag{1.20}$$

with initial condition $p(x,0)=\delta(x)$. We extend this result to the case where the Brownian motion is substituted by the fractional Brownian motion $B_H(t), t>0$, with Hurst parameter $H\in(0,1)$. We show that the density function of $B_H(|C(t)|), t>0$ resolves the following equation

$$t^2\frac{\partial^2 p}{\partial t^2}=-\left[H(H-1)\frac{\partial}{\partial x}x-H^2\frac{\partial^2}{\partial x^2}x^2\right]p-\frac{2Ht}{\pi}\frac{d^2}{dx^2}\delta(x)1_{\left\{0<H\leq\frac{1}{2}\right\}},$$

with $p(x,0)=\delta(x)$.

Also the case of a Cauchy process with a random time represented by an iterated Brownian motion is taken into account and the corresponding fractional governing equation is derived.

The iterated Cauchy process, defined as $J_{CC}(t)=C_1(|C_2(t)|), t>0$ has been considered in D'Ovidio and Orsingher (2009) and its connection with the wave equation

$$\frac{\partial^2 p}{\partial t^2}=\frac{\partial^2 p}{\partial x^2}-\frac{1}{\pi t x^2}, \qquad x\in\mathbb{R},\ t>0 \tag{1.21}$$

has been established.

We will show here that when the process C_1 is endowed with a position parameter $a \neq 0$, the distribution of the iterated Cauchy process satisfies the following wave equation

$$\frac{\partial^2 p}{\partial t^2} = \frac{\partial^2 p}{\partial x^2} - \frac{1}{\pi t (x-a)^2}, \qquad x \in \mathbb{R},\ t > 0. \tag{1.22}$$

Finally we consider the d-dimensional vector $(C_1(|C(t)|), ..., C_d(|C(t)|))$ and obtain the equation governed by its density, i.e.

$$p_{CC}^d(x_1, ..., x_d, t) = \frac{2}{\pi^{d+1}} \int_0^{+\infty} \prod_{j=1}^d \frac{s}{s^2 + x_j^2} \frac{t}{t^2 + s^2} ds, \tag{1.23}$$

where the use of partial Riesz fractional derivatives is required.

2 Iterated Brownian motion with drift

If the processes composing the iterated Brownian motions possess drift, the fractional equations and the higher-order equations governing the distributions are somewhat different. We start by considering the process $I^\mu(t) = B_1^\mu(|B_2(t)|), t > 0$, with law

$$p_I^\mu(x, t) = 2 \int_0^\infty \frac{e^{-\frac{(x-\mu s)^2}{2s}}}{\sqrt{2\pi s}} \frac{e^{-\frac{s^2}{2t}}}{\sqrt{2\pi t}} ds. \tag{2.1}$$

It has been shown in Beghin and Orsingher (2009) that (2.1) solves the following fractional equation of order $\nu = 1/2$:

$$\frac{\partial^{\frac{1}{2}} p}{\partial t^{\frac{1}{2}}} = \frac{1}{2^{3/2}} \frac{\partial^2 p}{\partial x^2} - \frac{\mu}{\sqrt{2}} \frac{\partial p}{\partial x}, \qquad x \in \mathbb{R},\ t > 0. \tag{2.2}$$

As a check we evaluate the Laplace-Fourier transform of (2.1):

$$\begin{aligned} & \int_0^\infty e^{-\eta t} \int_{-\infty}^{+\infty} e^{i\beta x} \left(2 \int_0^\infty \frac{e^{-\frac{(x-\mu s)^2}{2s}}}{\sqrt{2\pi s}} \frac{e^{-\frac{s^2}{2t}}}{\sqrt{2\pi t}} ds \right) dx dt \qquad (2.3) \\ = \;& 2 \int_0^\infty e^{-\frac{1}{2}\beta^2 s + i\beta\mu s} \frac{e^{-s\sqrt{2\eta}}}{\sqrt{2\eta}} ds = \frac{\eta^{\frac{1}{2}-1}}{\frac{\beta^2}{2^{3/2}} - \frac{i\beta\mu}{\sqrt{2}} + \sqrt{\eta}}. \end{aligned}$$

The previous result coincides with the Fourier-Laplace transform of (2.2).

We show in the following theorem that, as in the case where $\mu = 0$, the density of $I^\mu(t)$ satisfies also a fourth-order equation.

Theorem 2.1 *The density of the process $I^\mu(t) = B_1^\mu(|B_2(t)|), t > 0$ given in (2.1) is a solution to the following equation*

$$\frac{\partial}{\partial t} p(x,t) = \frac{1}{2} \left(\frac{1}{2} \frac{\partial^2}{\partial x^2} - \mu \frac{\partial}{\partial x} \right)^2 p(x,t) + \frac{1}{\sqrt{2\pi t}} \left(\frac{1}{2} \frac{d^2}{dx^2} - \mu \frac{d}{dx} \right) \delta(x), \qquad x \in \mathbb{R},\ t > 0 \tag{2.4}$$

with initial condition $p(x,0)=\delta(x)$.

Proof By taking the first-order derivative of (2.1), we have that

$$\begin{aligned}
&\frac{\partial}{\partial t}p_I^\mu(x,t) &(2.5)\\
=\;& 2\int_0^\infty \frac{e^{-\frac{(x-\mu s)^2}{2s}}}{\sqrt{2\pi s}}\frac{\partial}{\partial t}\frac{e^{-\frac{s^2}{2t}}}{\sqrt{2\pi t}}ds\\
=\;& \int_0^\infty \frac{e^{-\frac{(x-\mu s)^2}{2s}}}{\sqrt{2\pi s}}\frac{\partial^2}{\partial s^2}\frac{e^{-\frac{s^2}{2t}}}{\sqrt{2\pi t}}ds\\
=\;& [\text{integrating by parts}]\\
=\;& -\int_0^\infty \frac{\partial}{\partial s}\frac{e^{-\frac{(x-\mu s)^2}{2s}}}{\sqrt{2\pi s}}\frac{\partial}{\partial s}\frac{e^{-\frac{s^2}{2t}}}{\sqrt{2\pi t}}ds\\
=\;& -\frac{\partial}{\partial s}\frac{e^{-\frac{(x-\mu s)^2}{2s}}}{\sqrt{2\pi s}}\frac{e^{-\frac{s^2}{2t}}}{\sqrt{2\pi t}}\Bigg|_0^\infty+\int_0^\infty \frac{\partial^2}{\partial s^2}\frac{e^{-\frac{(x-\mu s)^2}{2s}}}{\sqrt{2\pi s}}\frac{e^{-\frac{s^2}{2t}}}{\sqrt{2\pi t}}ds\\
=\;& \frac{1}{\sqrt{2\pi t}}\left(\frac{1}{2}\frac{d^2}{dx^2}-\mu\frac{d}{dx}\right)\delta(x)+\int_0^\infty \frac{\partial}{\partial s}\left(\frac{1}{2}\frac{\partial^2}{\partial x^2}-\mu\frac{\partial}{\partial x}\right)\frac{e^{-\frac{(x-\mu s)^2}{2s}}}{\sqrt{2\pi s}}\frac{e^{-\frac{s^2}{2t}}}{\sqrt{2\pi t}}ds\\
=\;& \frac{1}{2}\left(\frac{1}{2}\frac{\partial^2}{\partial x^2}-\mu\frac{\partial}{\partial x}\right)^2 p_I^\mu(x,t)+\frac{1}{\sqrt{2\pi t}}\left(\frac{1}{2}\frac{d^2}{dx^2}-\mu\frac{d}{dx}\right)\delta(x).
\end{aligned}$$

□

Remark 2.1 The differential operator appearing in the fourth-order equation (2.4) is the formal square of the operator appearing in the fractional equation (2.2). In the special case where $\mu=0$, we obtain again equation (1.3).

We consider now the case where the process representing the "time" possesses drift, then the law of the iterated process $I^\mu(t)=B_1(|B_2^\mu(t)|), t>0$, reads

$$q_I^\mu(x,t)=\frac{1}{C(t)}\int_0^\infty \frac{e^{-\frac{x^2}{2s}}}{\sqrt{2\pi s}}\frac{e^{-\frac{(s-\mu t)^2}{2t}}}{\sqrt{2\pi t}}ds, \tag{2.6}$$

where

$$C(t)=\Pr\{B_2^\mu(t)>0\}=\int_{-\mu\sqrt{t}}^\infty \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}dy$$

is the normalizing factor for the density of $|B_2^\mu(t)|$.

The Fourier transform of $q_I^\mu(x,t)$ becomes

$$\int_{-\infty}^{+\infty} e^{i\beta x}q_I^\mu(x,t)dx=\frac{1}{C(t)}\int_{-\mu\sqrt{t}}^\infty \frac{e^{-\frac{\beta^2}{2}(\sqrt{t}w+\mu t)-\frac{w^2}{2}}}{\sqrt{2\pi}}dw. \tag{2.7}$$

The evaluation of the Laplace transform of (2.7) poses serious problems. For this reason the case where the process representing the "time" possesses drift is not further developed here.

3 Iterated processes involving the Cauchy process

The transition function of a centered Cauchy process $C(t), t > 0$ is given by

$$p_C(x,t) = \frac{t}{\pi(t^2+x^2)}, \quad x \in \mathbb{R},\ t > 0 \tag{3.1}$$

and it is well-known that p_C is a solution to the Laplace equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial t^2}\right) p_C(x,t) = 0. \tag{3.2}$$

Furthermore p_C is a solution to the following space-fractional equation

$$\frac{\partial}{\partial t} p_C(x,t) = -\frac{\partial}{\partial |x|} p_C(x,t), \tag{3.3}$$

where

$$\frac{\partial}{\partial |x|} f(x) = \frac{1}{\pi}\frac{d}{dx}\left[\int_{-\infty}^{x} \frac{f(y)}{x-y} dy - \int_{x}^{\infty} \frac{f(y)}{y-x} dy\right] \tag{3.4}$$

is a special case (for $\nu = 1$) of the Riesz fractional space-derivative and possesses Fourier transform equal to

$$\int_{-\infty}^{+\infty} e^{i\beta x} \frac{\partial}{\partial |x|} f(x) dx = |\beta| \int_{-\infty}^{+\infty} e^{i\beta x} f(x) dx = |\beta| \mathcal{F}(\beta). \tag{3.5}$$

By composing the standard Brownian motion and the Cauchy process, we obtain the following new processes:

$$J_{BC}(t) = B(|C(t)|), \qquad t > 0 \tag{3.6}$$

and

$$J_{CB}(t) = C(|B(t)|), \qquad t > 0, \tag{3.7}$$

where C and B are mutually independent.

An alternative definition of the first-order fractional derivative is given in Saichev and Zaslavsky (1997) (see formula (A.39)) and reads

$$\frac{d}{d|x|} f(x) = -\frac{1}{\pi} \int_0^{+\infty} \frac{f(x-y) - 2f(x) + f(x+y)}{y^2} dy. \tag{3.8}$$

This is a special case of

$$\frac{d^\nu}{d|x|^\nu} f(x) = \frac{1}{2\Gamma(-\nu)\cos\frac{\pi\nu}{2}} \int_0^{+\infty} \frac{f(x-y) - 2f(x) + f(x+y)}{y^2} dy, \tag{3.9}$$

for $0 < \nu \leq 1$. We note that (3.9) reduces to (3.8) for $\nu = 1$, because, by the reflection formula of the Gamma function, the constant can be rewritten as

$$\begin{aligned} \frac{1}{2\Gamma(-\nu)\cos\frac{\pi\nu}{2}} &= -\frac{1}{2\cos\frac{\pi\nu}{2}} \frac{\Gamma(1+\nu)}{\frac{\pi}{\sin \pi\nu}} \\ &= -\frac{\Gamma(1+\nu)\sin\frac{\pi\nu}{2}}{\pi}, \end{aligned}$$

which, for $\nu = 1$, yields $-1/\pi$.

We can convert definition (3.4) into (3.8) by means of integrations by parts, after suitable changes of variables.

As a further check we can show that the Fourier transform of (3.8) coincides with (3.5):

$$\begin{aligned}
& -\frac{1}{\pi}\int_{-\infty}^{+\infty} e^{i\beta x}\left(\int_0^{+\infty}\frac{f(x-y)-2f(x)+f(x+y)}{y^2}dy\right)dx \qquad (3.10)\\
= \; & -\frac{1}{\pi}\int_{-\infty}^{+\infty} f(w)e^{i\beta w}dw\int_0^{+\infty}\frac{e^{i\beta y}}{y^2}dy + \frac{2}{\pi}\int_{-\infty}^{+\infty} f(x)e^{i\beta x}dx\int_0^{+\infty}\frac{1}{y^2}dy + \\
& -\frac{1}{\pi}\int_{-\infty}^{+\infty} f(w)e^{i\beta w}dw\int_0^{+\infty}\frac{e^{-i\beta y}}{y^2}dy \\
= \; & -\frac{1}{\pi}\mathcal{F}(\beta)\int_0^{+\infty}\frac{e^{i\beta y}-2+e^{-i\beta y}}{y^2}dy.
\end{aligned}$$

The last integral can be evaluated as follows:

$$\begin{aligned}
& \int_0^{+\infty}\frac{e^{i\beta y}-1}{y^2}dy + \int_0^{+\infty}\frac{e^{-i\beta y}-1}{y^2}dy \\
= \; & \int_0^{+\infty}\frac{1}{y^2}dy\int_0^{y} i\beta e^{i\beta w}dw - \int_0^{+\infty}\frac{1}{y^2}dy\int_0^{y} i\beta e^{-i\beta w}dw \\
= \; & i\beta\int_0^{+\infty} e^{i\beta w}dw\int_w^{+\infty}\frac{1}{y^2}dy - i\beta\int_0^{+\infty} e^{-i\beta w}dw\int_w^{+\infty}\frac{1}{y^2}dy \\
= \; & -\left[2\beta\int_0^{+\infty}\frac{e^{i\beta w}}{2wi}dw - 2\beta\int_0^{+\infty}\frac{e^{-i\beta w}}{2wi}dw\right] \\
= \; & -2\beta\int_0^{+\infty}\frac{\sin\beta w}{w}dw = -\pi|\beta|,
\end{aligned}$$

which, inserted into (3.10), gives (3.5).

It has been shown in D'Ovidio and Orsingher (2009) that the transition density of the process $J_{BC}(t) = B(|C(t)|)$, $t > 0$, which is given by

$$p_{BC}(x,t) = \frac{2}{\pi}\int_0^{\infty}\frac{e^{-\frac{x^2}{2s}}}{\sqrt{2\pi s}}\frac{t}{t^2+s^2}ds, \quad x \in \mathbb{R},\ t > 0 \tag{3.11}$$

satisfies the following equation

$$\frac{\partial^2 p}{\partial t^2} = -\frac{1}{2^2}\frac{\partial^4 p}{\partial x^4} - \frac{1}{\pi t}\frac{d^2}{dx^2}\delta(x), \qquad x \in \mathbb{R},\ t > 0, \tag{3.12}$$

with initial condition $p(x, 0) = \delta(x)$.

Remark 3.1 In mathematical physics the equation

$$\frac{\partial^2 u}{\partial t^2} = -K\frac{\partial^4 u}{\partial x^4} \tag{3.13}$$

represents the vibration of rods (see Elmore and Heald (1969), p.116). Equation (3.13) coincides with (3.12) for $x \neq 0$.

Remark 3.2 The density (3.11) can be worked out in an equivalent form, by applying the subordinating relationship

$$\frac{t}{\pi(t^2+s^2)} = \int_0^\infty \frac{e^{-\frac{s^2}{2w}}}{\sqrt{2\pi w}} \frac{te^{-\frac{t^2}{2w}}}{\sqrt{2\pi w^3}} dw.$$

Thus the transition density of the process $I_{BC}(t) = B(|C(t)|)$, $t > 0$ can be rewritten as

$$p_{BC}(x,t) = 2\int_0^\infty \frac{e^{-\frac{x^2}{2s}}}{\sqrt{2\pi s}} \left(\int_0^\infty \frac{e^{-\frac{s^2}{2w}}}{\sqrt{2\pi w}} \frac{te^{-\frac{t^2}{2w}}}{\sqrt{2\pi w^3}} dw \right) ds. \tag{3.14}$$

Formula (3.14) corresponds to the law of an iterated Brownian motion $I(t) = B_1(|B_2(t)|)$, taken at a random time $T(t)$, which coincides in distribution with the first-passage time of a standard Brownian motion. Thus (3.14) coincides with the density of the process $I(T(t)), t > 0$, where $T(t) = \inf(s : B(s) = t)$.

We consider now the composition of a fractional Brownian motion with a Cauchy process, $J_{B_H C}(t) = B_H(|C(t)|), t > 0$.

Theorem 3.1 *The density of* $J_{B_H C}(t), t > 0$, *which is given by*

$$p_{B_H C}(x,t) = \frac{2}{\pi} \int_0^{+\infty} \frac{e^{-\frac{x^2}{2s^{2H}}}}{\sqrt{2\pi s^{2H}}} \frac{t}{t^2+s^2} ds, \tag{3.15}$$

satisfies the following equation

$$t^2 \frac{\partial^2 p}{\partial t^2} = -\left[H(H-1)\frac{\partial}{\partial x} x - H^2 \frac{\partial^2}{\partial x^2} x^2 \right] p - \frac{2Ht}{\pi} \frac{d^2}{dx^2} \delta(x) 1_{\{0<H\leq\frac{1}{2}\}}, \tag{3.16}$$

with initial condition $p(x,0) = \delta(x)$.

Proof We take the second order time-derivative of (3.15), that is

$$\begin{aligned} \frac{\partial^2}{\partial t^2} p_{B_H C}(x,t) &= \frac{2}{\pi} \int_0^{+\infty} \frac{e^{-\frac{x^2}{2s^{2H}}}}{\sqrt{2\pi s^{2H}}} \frac{\partial^2}{\partial t^2} \left(\frac{t}{t^2+s^2} \right) ds \qquad (3.17) \\ &= -\frac{2}{\pi} \int_0^{+\infty} \frac{e^{-\frac{x^2}{2s^{2H}}}}{\sqrt{2\pi s^{2H}}} \frac{\partial^2}{\partial s^2} \left(\frac{t}{t^2+s^2} \right) ds \\ &= \frac{2}{\pi} \frac{\partial}{\partial s} \left(\frac{e^{-\frac{x^2}{2s^{2H}}}}{\sqrt{2\pi s^{2H}}} \right) \frac{t}{t^2+s^2} \Bigg|_{s=0}^{s=+\infty} + \\ & \quad -\frac{2}{\pi} \int_0^{+\infty} \frac{\partial^2}{\partial s^2} \left(\frac{e^{-\frac{x^2}{2s^{2H}}}}{\sqrt{2\pi s^{2H}}} \right) \frac{t}{t^2+s^2} ds. \end{aligned}$$

The first term in (3.17) can be worked out by considering that the transition density of the fractional Brownian motion $p_H = p_H(x,t)$ satisfies the following equation

$$\frac{\partial p}{\partial t} = Ht^{2H-1}\frac{\partial^2 p}{\partial x^2}$$

and it becomes

$$-\frac{2H}{\pi}\frac{ts^{2H-1}}{t^2+s^2}\frac{\partial^2}{\partial x^2}\left(\frac{e^{-\frac{x^2}{2s^{2H}}}}{\sqrt{2\pi s^{2H}}}\right)\Bigg|_{s=0} = \begin{cases} 0 & \frac{1}{2}<H<1 \\ -\frac{2Ht}{\pi}\frac{d^2}{dx^2}\delta(x) & 0<H\leq\frac{1}{2} \end{cases}. \tag{3.18}$$

The second term in (3.17) can be evaluated as follows

$$\begin{aligned} & -\frac{2}{\pi}\int_0^{+\infty}\frac{\partial}{\partial s}\left(Hs^{2H-1}\frac{\partial^2}{\partial x^2}\left(\frac{e^{-\frac{x^2}{2s^{2H}}}}{\sqrt{2\pi s^{2H}}}\right)\right)\frac{t}{t^2+s^2}ds \qquad (3.19)\\ = & -\frac{2H(2H-1)}{\pi}\int_0^{+\infty}s^{2H-2}\frac{\partial^2}{\partial x^2}\left(\frac{e^{-\frac{x^2}{2s^{2H}}}}{\sqrt{2\pi s^{2H}}}\right)\frac{t}{t^2+s^2}ds + \\ & -\frac{2H^2}{\pi}\int_0^{+\infty}s^{4H-2}\frac{\partial^4}{\partial x^4}\left(\frac{e^{-\frac{x^2}{2s^{2H}}}}{\sqrt{2\pi s^{2H}}}\right)\frac{t}{t^2+s^2}ds \\ = & +\frac{2H(2H-1)}{\pi}\frac{\partial}{\partial x}\left(x\int_0^{+\infty}s^{-2}\frac{e^{-\frac{x^2}{2s^{2H}}}}{\sqrt{2\pi s^{2H}}}\frac{t}{t^2+s^2}ds\right) + \\ & +\frac{2H^2}{\pi}\int_0^{+\infty}s^{2H-2}\frac{\partial^3}{\partial x^3}\left(x\frac{e^{-\frac{x^2}{2s^{2H}}}}{\sqrt{2\pi s^{2H}}}\right)\frac{t}{t^2+s^2}ds. \end{aligned}$$

We rewrite the last line in (3.19) as

$$\begin{aligned} & \frac{2H^2}{\pi}\int_0^{+\infty}s^{2H-2}\frac{\partial^2}{\partial x^2}\left(\frac{e^{-\frac{x^2}{2s^{2H}}}}{\sqrt{2\pi s^{2H}}}\right)\frac{t}{t^2+s^2}ds + \qquad (3.20)\\ & -\frac{2H^2}{\pi}\int_0^{+\infty}s^{-2}\frac{\partial^2}{\partial x^2}\left(x^2\frac{e^{-\frac{x^2}{2s^{2H}}}}{\sqrt{2\pi s^{2H}}}\right)\frac{t}{t^2+s^2}ds \\ = & -\frac{2H^2}{\pi}\int_0^{+\infty}s^{-2}\frac{\partial}{\partial x}\left(x\frac{e^{-\frac{x^2}{2s^{2H}}}}{\sqrt{2\pi s^{2H}}}\right)\frac{t}{t^2+s^2}ds + \\ & -\frac{2H^2}{\pi}\int_0^{+\infty}s^{-2}\frac{\partial^2}{\partial x^2}\left(x^2\frac{e^{-\frac{x^2}{2s^{2H}}}}{\sqrt{2\pi s^{2H}}}\right)\frac{t}{t^2+s^2}ds. \end{aligned}$$

By inserting (3.20) into (3.19) the second term of (3.17) takes the form

$$\left[\frac{2H(H-1)}{\pi}\frac{\partial}{\partial x}x - \frac{2H^2}{\pi}\frac{\partial^2}{\partial x^2}x^2\right]\int_0^{+\infty}s^{-2}\frac{e^{-\frac{x^2}{2s^{2H}}}}{\sqrt{2\pi s^{2H}}}\frac{t}{t^2+s^2}ds. \tag{3.21}$$

The integral appearing in (3.21) can be calculated by observing that

$$
\begin{aligned}
&\frac{1}{t}\int_0^{+\infty} s^{-2}\frac{e^{-\frac{x^2}{2s^{2H}}}}{\sqrt{2\pi s^{2H}}}\frac{t^2+s^2-s^2}{t^2+s^2}ds \\
= \quad & -\frac{1}{t^2}\frac{\pi}{2}p_{B_H C}(x,t)+\frac{1}{t}\int_0^{+\infty} s^{-2}\frac{e^{-\frac{x^2}{2s^{2H}}}}{\sqrt{2\pi s^{2H}}}ds.
\end{aligned} \tag{3.22}
$$

The integral in (3.22) can be explicitly calculated as

$$
\begin{aligned}
&\int_0^{+\infty} s^{-2}\frac{e^{-\frac{x^2}{2s^{2H}}}}{\sqrt{2\pi s^{2H}}}ds \\
= \quad & \left[s=\frac{x^{1/H}}{2^{1/2H}}y^{-1/2H}\right] \\
= \quad & \int_0^{+\infty} \frac{e^{-y}x^{-2/H}y^{1/H}}{2^{-1/H}\sqrt{2\pi}}\frac{x^{1/H}}{2^{1/2H}}\left(-\frac{1}{2H}\right)y^{-\frac{1}{2H}-1}\frac{(2y)^{1/2}}{x}dy \\
= \quad & \frac{2^{\frac{1}{2H}-1}}{x^{\frac{1}{H}+1}H\sqrt{\pi}}\Gamma\left(\frac{1}{2H}+\frac{1}{2}\right),
\end{aligned}
$$

which, inserted in (3.22), permits us to rewrite (3.21) as follows:

$$
\begin{aligned}
&\left[\frac{2H(H-1)}{\pi}\frac{\partial}{\partial x}x-\frac{2H^2}{\pi}\frac{\partial^2}{\partial x^2}x^2\right]\left(-\frac{1}{t^2}\frac{\pi}{2}p_{B_H C}+\frac{2^{\frac{1}{2H}-1}}{x^{\frac{1}{H}+1}H\sqrt{\pi}t}\Gamma\left(\frac{1}{2H}+\frac{1}{2}\right)\right) \\
= \quad & -\frac{1}{t^2}\left[H(H-1)\frac{\partial}{\partial x}x-H^2\frac{\partial^2}{\partial x^2}x^2\right]p_{B_H C}+ \\
& +\frac{2^{\frac{1}{2H}}(H-1)}{\pi\sqrt{\pi}t}\Gamma\left(\frac{1}{2H}+\frac{1}{2}\right)\frac{d}{dx}\left(x^{-\frac{1}{H}}\right)-\frac{2^{\frac{1}{2H}}H}{\pi\sqrt{\pi}t}\Gamma\left(\frac{1}{2H}+\frac{1}{2}\right)\frac{d^2}{dx^2}\left(x^{-\frac{1}{H}+1}\right) \\
= \quad & -\frac{1}{t^2}\left[H(H-1)\frac{\partial}{\partial x}x-H^2\frac{\partial^2}{\partial x^2}x^2\right]p_{B_H C}+ \\
& -\frac{2^{\frac{1}{2H}}(H-1)}{\pi\sqrt{\pi}tH}\Gamma\left(\frac{1}{2H}+\frac{1}{2}\right)x^{-\frac{1}{H}-1}+\frac{2^{\frac{1}{2H}}(H-1)}{\pi\sqrt{\pi}tH}\Gamma\left(\frac{1}{2H}+\frac{1}{2}\right)x^{-\frac{1}{H}-1} \\
= \quad & -\frac{1}{t^2}\left[H(H-1)\frac{\partial}{\partial x}x-H^2\frac{\partial^2}{\partial x^2}x^2\right]p_{B_H C}.
\end{aligned} \tag{3.23}
$$

By putting together (3.23) and (3.18) we obtain equation (3.16). □

Remark 3.3 In order to check that for $H=\frac{1}{2}$ equation (3.16) can be converted into (3.12), it is

necessary to perform the following calculations:

$$\begin{aligned}
& \frac{\partial^4}{\partial x^4}\left(\frac{2}{\pi}\int_0^{+\infty}\frac{e^{-\frac{x^2}{2s}}}{\sqrt{2\pi s}}\frac{t}{t^2+s^2}ds\right) \qquad (3.24)\\
= & -\frac{\partial^3}{\partial x^3}\left(\frac{2}{\pi}x\int_0^{+\infty}\frac{e^{-\frac{x^2}{2s}}}{s\sqrt{2\pi s}}\frac{t}{t^2+s^2}ds\right)\\
= & -\frac{2}{\pi}\frac{\partial^2}{\partial x^2}\left(\int_0^{+\infty}\frac{e^{-\frac{x^2}{2s}}}{s\sqrt{2\pi s}}\frac{t}{t^2+s^2}ds - x^2\int_0^{+\infty}\frac{1}{s^2}\frac{e^{-\frac{x^2}{2s}}}{\sqrt{2\pi s}}\frac{t}{t^2+s^2}ds\right).
\end{aligned}$$

In analogy with (3.22), we can write

$$\int_0^{+\infty}\frac{1}{s^2}\frac{e^{-\frac{x^2}{2s}}}{\sqrt{2\pi s}}\frac{t}{t^2+s^2}ds = -\frac{1}{t^2}\frac{\pi}{2}p_{BC}(x,t)+\frac{1}{t}\int_0^{+\infty}\frac{1}{s^2}\frac{e^{-\frac{x^2}{2s}}}{\sqrt{2\pi s}}ds. \qquad (3.25)$$

Furthermore

$$\begin{aligned}
& \frac{\partial^2}{\partial x^2}\left(\int_0^{+\infty}\frac{e^{-\frac{x^2}{2s}}}{s\sqrt{2\pi s}}\frac{t}{t^2+s^2}ds\right) \qquad (3.26)\\
= & -\frac{\partial}{\partial x}\left(x\int_0^{+\infty}\frac{e^{-\frac{x^2}{2s}}}{s^2\sqrt{2\pi s}}\frac{t}{t^2+s^2}ds\right)\\
= & -\frac{\partial}{\partial x}\left[x\left(-\frac{1}{t^2}\frac{\pi}{2}p_{BC}(x,t)+\frac{1}{t}\int_0^{+\infty}\frac{1}{s^2}\frac{e^{-\frac{x^2}{2s}}}{\sqrt{2\pi s}}ds\right)\right].
\end{aligned}$$

By inserting (3.25) and (3.26) into (3.24) we have that

$$\begin{aligned}
\frac{\partial^4}{\partial x^4}p_{BC}(x,t) = & -\frac{1}{t^2}\frac{\partial^2}{\partial x^2}\left(x^2 p_{BC}(x,t)\right)-\frac{1}{t^2}\frac{\partial}{\partial x}\left(x\,p_{BC}(x,t)\right)+\\
& +\frac{2}{\pi t}\frac{\partial^2}{\partial x^2}\left(x^2\int_0^{+\infty}\frac{e^{-\frac{x^2}{2s}}}{s^2\sqrt{2\pi s}}ds\right)+\frac{2}{\pi t}\frac{\partial}{\partial x}\left(x\int_0^{+\infty}\frac{e^{-\frac{x^2}{2s}}}{s^2\sqrt{2\pi s}}ds\right)\\
= & -\frac{1}{t^2}\frac{\partial^2}{\partial x^2}\left(x^2 p_{BC}(x,t)\right)-\frac{1}{t^2}\frac{\partial}{\partial x}\left(x\,p_{BC}(x,t)\right)+\\
& +\frac{2}{\pi t}\left[\frac{\partial^2}{\partial x^2}x^2+\frac{\partial}{\partial x}x\right]\int_0^{+\infty}\frac{e^{-\frac{x^2}{2s}}}{s^2\sqrt{2\pi s}}ds.
\end{aligned}$$

The calculations carried out in Theorem 3.1 permit us to conclude that, for $x \neq 0$,

$$\frac{\partial^4}{\partial x^4}p_{BC}(x,t) = -\frac{1}{t^2}\frac{\partial^2}{\partial x^2}\left(x^2 p_{BC}(x,t)\right)-\frac{1}{t^2}\frac{\partial}{\partial x}\left(x\,p_{BC}(x,t)\right) = -\frac{\partial^2}{\partial t^2}p_{BC}(x,t).$$

Therefore, despite the formal diversity of (3.12) and (3.16), the law of $B(|C(t)|)$ satisfies also equation (3.16).

We consider now the d-dimensional generalization of the results given in (3.11) and (3.12): let us define the vector process

$$J^d_{BC}(t) = \begin{cases} B_1(|C(t)|) \\ \cdots \\ \cdots \\ B_d(|C(t)|) \end{cases}, \qquad t > 0,$$

where $B_1, ..., B_d$ are Brownian motions independent from each other and from the Cauchy process $C(t), t > 0$. Then the following result holds.

Theorem 3.2 *The joint density of* $J^d_{BC}(t), t > 0$, *which is given by*

$$p^d_{BC}(x_1, ..., x_d, t) = \frac{2}{\pi} \int_0^{+\infty} \prod_{j=1}^d \frac{e^{-\frac{x_j^2}{2s}}}{\sqrt{2\pi s}} \frac{t}{t^2 + s^2} ds, \tag{3.27}$$

is a solution to the following equation

$$\frac{\partial^2 p}{\partial t^2} = -\frac{1}{2^2} \left(\sum_{j=1}^d \frac{\partial^2}{\partial x_j^2} \right)^2 p(x_1, ..., x_d, t) - \frac{1}{\pi t} \sum_{j=1}^d \delta(x_1)...\frac{d^2}{dx_j^2}\delta(x_j)...\delta(x_d), \tag{3.28}$$

with initial condition $p(x_1, ..., x_d, 0) = \prod_{j=1}^d \delta(x_j)$.
Proof By taking the second order time-derivative of (3.27) we get, by means of two integrations by parts, that

$$\begin{aligned} & \frac{\partial^2}{\partial t^2} p^d_{BC}(x_1, ..., x_d, t) \qquad (3.29) \\ = & -\frac{2}{\pi} \int_0^{+\infty} \frac{\partial^2}{\partial s^2} \left(\prod_{j=1}^d \frac{e^{-\frac{x_j^2}{2s}}}{\sqrt{2\pi s}} \right) \frac{t}{t^2 + s^2} ds + \\ & + \frac{2}{\pi} \frac{\partial}{\partial s} \left(\prod_{j=1}^d \frac{e^{-\frac{x_j^2}{2s}}}{\sqrt{2\pi s}} \right) \frac{t}{t^2 + s^2} \Bigg|_0^{+\infty} \\ = & -\frac{2}{\pi} \left(\frac{1}{2} \sum_{j=1}^d \frac{\partial^2}{\partial x_j^2} \right)^2 \int_0^{+\infty} \prod_{j=1}^d \frac{e^{-\frac{x_j^2}{2s}}}{\sqrt{2\pi s}} \frac{t}{t^2 + s^2} ds + \\ & + \frac{1}{\pi} \frac{t}{t^2 + s^2} \sum_{j=1}^d \frac{\partial^2}{\partial x_j^2} \left(\prod_{j=1}^d \frac{e^{-\frac{x_j^2}{2s}}}{\sqrt{2\pi s}} \right) \Bigg|_0^{+\infty}, \end{aligned}$$

which coincides with (3.28). □

Remark 3.4 In view of Remark 3.2, we note that the vector process $J^d_{BC}(t)$ is equivalent in distribution to the following d-dimensional process:

$$\begin{cases} I_1(T(t)) \\ \quad ... \\ \quad ... \\ I_d(T(t)) \end{cases}, \qquad t>0,$$

where $I_j,\ j=1,...n$ are independent iterated Brownian motions and $T(t)$ coincides with the first-passage time of a standard Brownian motion.

We focus now our attention on the process

$$J_{CB}(t)=C(|B(t)|), \qquad t>0,$$

and prove that its density

$$p_{CB}(x,t)=\frac{2}{\pi}\int_0^{+\infty}\frac{s}{s^2+x^2}\frac{e^{-\frac{s^2}{2t}}}{\sqrt{2\pi t}}ds \tag{3.30}$$

satisfies a non-homogeneous backward heat-equation.

Theorem 3.3 *The density (3.30) is a solution to the following equation*

$$\frac{\partial p}{\partial t}=-\frac{1}{2}\frac{\partial^2 p}{\partial x^2}+\frac{1}{\pi x^2\sqrt{2\pi t}}, \qquad x\in\mathbb{R},\ t>0, \tag{3.31}$$

with initial condition $p(x,0)=\delta(x)$.

Proof By taking the time-derivative of $p_{CB}=p_{CB}(x,t)$, we have that

$$\begin{aligned}\frac{\partial p}{\partial t} &= \frac{2}{\pi}\int_0^{+\infty}\frac{s}{s^2+x^2}\frac{\partial}{\partial t}\left(\frac{e^{-\frac{s^2}{2t}}}{\sqrt{2\pi t}}\right)ds \\ &= \frac{1}{\pi}\int_0^{+\infty}\frac{s}{s^2+x^2}\frac{\partial^2}{\partial s^2}\left(\frac{e^{-\frac{s^2}{2t}}}{\sqrt{2\pi t}}\right)ds \\ &= -\frac{1}{\pi}\int_0^{+\infty}\frac{\partial}{\partial s}\left(\frac{s}{s^2+x^2}\right)\frac{e^{-\frac{s^2}{2t}}}{\sqrt{2\pi t}}\Bigg|_0^{+\infty}+ \\ &\quad +\frac{1}{\pi}\int_0^{+\infty}\frac{\partial^2}{\partial s^2}\left(\frac{s}{s^2+x^2}\right)\frac{e^{-\frac{s^2}{2t}}}{\sqrt{2\pi t}}ds,\end{aligned}$$

which coincides with (3.31). □

The previous results can be further extended to the case where the "time process" is represented by a n-times iterated, instead of standard, Brownian motion. We prove that the density of the process

$$C(|I_n(t)|)=C(|B_1(|B_2(...|B_{n+1}(t)|...)|)|), \qquad t>0$$

is a solution to a non-homogeneous fractional equation.

Theorem 3.4 *The density of the process $C(|I_n(t)|), t>0$, which is given by*

$$p_{CI}(x,t) = \frac{2}{\pi}\int_0^{+\infty} \frac{s}{s^2+x^2} p_n(s,t)ds, \tag{3.32}$$

where

$$p_n(x,t) = 2^n \int_0^{+\infty} ... \int_0^{+\infty} \frac{e^{-\frac{x^2}{2w_1}}}{\sqrt{2\pi w_1}} dw_1 \frac{e^{-\frac{w_1^2}{2w_2}}}{\sqrt{2\pi w_2}} dw_2 ... \frac{e^{-\frac{w_n^2}{2t}}}{\sqrt{2\pi t}} dw_n,$$

satisfies the following equation

$$\frac{\partial^{\frac{1}{2^n}} p}{\partial t^{\frac{1}{2^n}}} = -2^{\frac{1}{2^n}-2} \frac{\partial^2 p}{\partial x^2} + \frac{2^{n-1+\frac{1}{2^{n+1}}}}{\pi^{3/2}} \frac{\Gamma\left(-\frac{1}{2}\right)}{\Gamma\left(-\frac{1}{2^{n+1}}\right)} \frac{t^{-\frac{1}{2^{n+1}}}}{x^2}, \qquad x\in\mathbb{R},\ t>0, \tag{3.33}$$

with initial condition $p(x,0)=\delta(x)$.
Proof We start by taking the fractional time-derivative of order $1/2^n$ of the density (3.32):

$$\begin{aligned}
& \frac{\partial^{\frac{1}{2^n}}}{\partial t^{\frac{1}{2^n}}} p_{CI}(x,t) \qquad (3.34)\\
= \;& \frac{2}{\pi}\int_0^{+\infty} \frac{s}{s^2+x^2} \frac{\partial^{\frac{1}{2^n}}}{\partial t^{\frac{1}{2^n}}} p_n(s,t)ds \\
= \;& \frac{2^{\frac{1}{2^n}-1}}{\pi}\int_0^{+\infty} \frac{s}{s^2+x^2} \frac{\partial^2}{\partial s^2} p_n(s,t)ds \\
= \;& \frac{2^{\frac{1}{2^n}-1}}{\pi} \frac{s}{s^2+x^2} \frac{\partial}{\partial s} p_n(s,t)\Big|_0^{+\infty} + \\
& - \frac{2^{\frac{1}{2^n}-1}}{\pi}\int_0^{+\infty} \frac{\partial}{\partial s}\left(\frac{s}{s^2+x^2}\right) \frac{\partial}{\partial s} p_n(s,t)ds \\
= \;& -\frac{2^{\frac{1}{2^n}-1}}{\pi} \frac{\partial}{\partial s}\left(\frac{s}{s^2+x^2}\right) p_n(s,t)\Big|_0^{+\infty} + \\
& + \frac{2^{\frac{1}{2^n}-1}}{\pi}\int_0^{+\infty} \frac{\partial^2}{\partial s^2}\left(\frac{s}{s^2+x^2}\right) p_n(s,t)ds \\
= \;& \frac{2^{\frac{1}{2^n}-1}}{\pi} \frac{1}{x^2} p_n(0,t) - 2^{\frac{1}{2^n}-2} \frac{\partial^2}{\partial x^2} p_{CI}(x,t).
\end{aligned}$$

We concentrate now on the first term and evaluate

$$
\begin{aligned}
p_n(0,t) &= 2^n \int_0^{+\infty} \frac{e^{-\frac{w_1^2}{2w_2}}}{\sqrt{2\pi w_1}} dw_1 \int_0^{+\infty} \frac{e^{-\frac{w_2^2}{2w_3}}}{\sqrt{2\pi w_2}} dw_2 ... \int_0^{+\infty} \frac{e^{-\frac{w_n^2}{2t}}}{\sqrt{2\pi w_n}\sqrt{2\pi t}} dw_n \\
&= \frac{2^{n-\frac{3}{4}}\Gamma\left(\frac{1}{4}\right)}{\left(\sqrt{2\pi}\right)^{n+1}} \int_0^{+\infty} w_2^{-\frac{1}{4}} e^{-\frac{w_2^2}{2w_3}} dw_2 ... \int_0^{+\infty} \frac{e^{-\frac{w_n^2}{2t}}}{\sqrt{w_n t}} dw_n \\
&= \frac{2^n}{\left(\sqrt{2\pi}\right)^{n+1}} 2^{-\frac{1}{2}-\frac{1}{4}}\Gamma\left(\frac{1}{4}\right) 2^{-\frac{1}{2}-\frac{1}{8}}\Gamma\left(\frac{3}{8}\right) ... 2^{-\frac{1}{2}-\frac{1}{2^n}}\Gamma\left(\frac{1}{2}-\frac{1}{2^n}\right) \int_0^{+\infty} \frac{w_n^{-\frac{1}{2^n}} e^{-\frac{w_n^2}{2t}}}{\sqrt{t}} dw_n \\
&= \frac{2^n}{\left(\sqrt{2\pi}\right)^{n+1}} 2^{-\frac{1}{2}-\frac{1}{4}}\Gamma\left(\frac{1}{4}\right) 2^{-\frac{1}{2}-\frac{1}{8}}\Gamma\left(\frac{3}{8}\right) ... 2^{-\frac{1}{2}-\frac{1}{2^n}}\Gamma\left(\frac{1}{2}-\frac{1}{2^n}\right) 2^{-\frac{1}{2}-\frac{1}{2^{n+1}}}\Gamma\left(\frac{1}{2}-\frac{1}{2^{n+1}}\right) t^{-\frac{1}{2^{n+1}}} \\
&= \frac{2^{-1+\frac{1}{2^{n+1}}}}{\pi^{\frac{n+1}{2}}} t^{-\frac{1}{2^{n+1}}} \Gamma\left(\frac{1}{2}-\frac{1}{2^2}\right)\Gamma\left(\frac{1}{2}-\frac{1}{2^3}\right) ... \Gamma\left(\frac{1}{2}-\frac{1}{2^n}\right)\Gamma\left(\frac{1}{2}-\frac{1}{2^{n+1}}\right) \\
&= \text{[by the duplication property of the Gamma function]} \\
&= \frac{2^{-1+\frac{1}{2^{n+1}}}}{\pi^{\frac{n+1}{2}}} t^{-\frac{1}{2^{n+1}}} \pi^{\frac{n}{2}} 2^{n+1-\frac{1}{2^n}} \frac{\Gamma\left(-\frac{1}{2}\right)}{\Gamma\left(-\frac{1}{2^{n+1}}\right)} \\
&= \frac{2^{n-\frac{1}{2^{n+1}}}}{\pi^{\frac{1}{2}}} t^{-\frac{1}{2^{n+1}}} \frac{\Gamma\left(-\frac{1}{2}\right)}{\Gamma\left(-\frac{1}{2^{n+1}}\right)},
\end{aligned}
$$

which, multiplied by the constant appearing in (3.34), gives the final form of equation (3.33). □

In the d-dimensional case we consider the vector process

$$
\begin{cases} C_1(|B(t)|) \\ ... \\ C_d(|B(t)|) \end{cases}, \qquad t > 0 \tag{3.35}
$$

and obtain the governing fractional equation in the following theorem.

Theorem 3.5 *The joint probability law of the process defined in (3.35) reads*

$$
p_{CB}^d(x_1, ..., x_d, t) = \frac{2}{\pi^d} \int_0^{+\infty} \prod_{j=1}^d \frac{s}{s^2 + x_j^2} \frac{e^{-\frac{s^2}{2t}}}{\sqrt{2\pi t}} ds \tag{3.36}
$$

and satisfies, for $d > 1$, the following fractional equation

$$
\frac{\partial p}{\partial t} = 2\sum_{k=2}^{d}\sum_{j=1}^{k-1} \frac{\partial^2 p}{\partial |x_k| \partial |x_j|} - \sum_{k=1}^{d} \frac{\partial^2 p}{\partial x_k^2}, \qquad x_j \in \mathbb{R},\ j = 1, ..., d,\ t > 0, \tag{3.37}
$$

with initial condition $p(x_1, ..., x_d, 0) = \prod_{j=1}^{d} \delta(x_j)$.

Proof The time-derivative of (3.36) can be evaluated as follows:

$$
\begin{aligned}
& \frac{\partial}{\partial t} p^d_{CB}(x_1, ..., x_d, t) && (3.38) \\
= \quad & \frac{2}{\pi^d} \int_0^{+\infty} \prod_{j=1}^{d} \frac{s}{s^2 + x_j^2} \frac{\partial}{\partial t} \left(\frac{e^{-\frac{s^2}{2t}}}{\sqrt{2\pi t}} \right) ds \\
= \quad & \frac{1}{\pi^d} \int_0^{+\infty} \prod_{j=1}^{d} \frac{s}{s^2 + x_j^2} \frac{\partial^2}{\partial s^2} \left(\frac{e^{-\frac{s^2}{2t}}}{\sqrt{2\pi t}} \right) ds \\
= \quad & -\frac{1}{\pi^d} \frac{\partial}{\partial s} \left(\prod_{j=1}^{d} \frac{s}{s^2 + x_j^2} \right) \frac{e^{-\frac{s^2}{2t}}}{\sqrt{2\pi t}} \Bigg|_0^{+\infty} + \frac{1}{\pi^d} \int_0^{+\infty} \frac{\partial^2}{\partial s^2} \left(\prod_{j=1}^{d} \frac{s}{s^2 + x_j^2} \right) \frac{e^{-\frac{s^2}{2t}}}{\sqrt{2\pi t}} ds \\
= \quad & -\frac{1}{\pi^d} \sum_{k=1}^{d} \prod_{\substack{j=1 \\ j \neq k}}^{d} \frac{s}{s^2 + x_j^2} \frac{\partial}{\partial s} \left(\frac{s}{s^2 + x_j^2} \right) \frac{e^{-\frac{s^2}{2t}}}{\sqrt{2\pi t}} \Bigg|_0^{+\infty} + \frac{1}{\pi^d} \int_0^{+\infty} \frac{\partial^2}{\partial s^2} \left(\prod_{j=1}^{d} \frac{s}{s^2 + x_j^2} \right) \frac{e^{-\frac{s^2}{2t}}}{\sqrt{2\pi t}} ds.
\end{aligned}
$$

The first term in the last member is equal to zero (unlike the one-dimensional case) and this makes equation (3.37) homogeneous.

Since

$$
\frac{\partial}{\partial s} \left(\prod_{j=1}^{d} \frac{s}{s^2 + x_j^2} \right) = \sum_{k=1}^{d} \prod_{\substack{j=1 \\ j \neq k}}^{d} \frac{s}{s^2 + x_j^2} \frac{\partial}{\partial s} \left(\frac{s}{s^2 + x_k^2} \right),
$$

the second-order derivative becomes

$$
\begin{aligned}
\frac{\partial^2}{\partial s^2} \left(\prod_{j=1}^{d} \frac{s}{s^2 + x_j^2} \right) \quad = \quad & \sum_{k=1}^{d} \left[\sum_{\substack{j=1 \\ j \neq k}}^{d} \frac{\partial}{\partial s} \left(\frac{s}{s^2 + x_k^2} \right) \frac{\partial}{\partial s} \left(\frac{s}{s^2 + x_j^2} \right) \prod_{\substack{l=1 \\ l \neq k,j}}^{d} \frac{s}{s^2 + x_l^2} + \right. \\
& \left. + \frac{\partial^2}{\partial s^2} \left(\frac{s}{s^2 + x_k^2} \right) \prod_{\substack{l=1 \\ l \neq k}}^{d} \frac{s}{s^2 + x_l^2} \right]. \qquad (3.39)
\end{aligned}
$$

In view of (3.2) and (3.3), we can rewrite (3.39) as

$$\begin{aligned}
\frac{\partial^2}{\partial s^2}\left(\prod_{j=1}^{d}\frac{s}{s^2+x_j^2}\right) &= \sum_{k=1}^{d}\left[\sum_{\substack{j=1\\j\neq k}}^{d}\frac{\partial}{\partial|x_k|}\left(\frac{s}{s^2+x_k^2}\right)\frac{\partial}{\partial|x_j|}\left(\frac{s}{s^2+x_j^2}\right)\prod_{\substack{l=1\\l\neq k,j}}^{d}\frac{s}{s^2+x_l^2}+\right.\\
&\quad\left.-\frac{\partial^2}{\partial x_k^2}\left(\frac{s}{s^2+x_k^2}\right)\prod_{\substack{l=1\\l\neq k}}^{d}\frac{s}{s^2+x_l^2}\right] \qquad (3.40)\\
&= \sum_{k=1}^{d}\sum_{\substack{j=1\\j\neq k}}^{d}\frac{\partial}{\partial|x_k|}\frac{\partial}{\partial|x_j|}\prod_{l=1}^{d}\frac{s}{s^2+x_l^2}-\sum_{k=1}^{d}\frac{\partial^2}{\partial x_k^2}\left(\prod_{l=1}^{d}\frac{s}{s^2+x_l^2}\right).
\end{aligned}$$

By inserting (3.40) in (3.38), we arrive at equation (3.37), by applying the commutative property of the fractional derivative (3.8). Therefore we show that

$$\frac{\partial}{\partial|x_j|}\frac{\partial}{\partial|x_k|}=\frac{\partial}{\partial|x_k|}\frac{\partial}{\partial|x_j|}=\frac{\partial^2}{\partial|x_k|\partial|x_j|}.$$

The second-order fractional derivative reads

$$\begin{aligned}
&\frac{\partial^2}{\partial|y|\partial|x|}f(x,y) \qquad (3.41)\\
=\;& -\frac{\partial}{\partial|y|}\left\{\frac{1}{\pi}\int_0^{+\infty}\frac{f(x-t,y)-2f(x,y)+f(x+t,y)}{t^2}dt\right\}\\
=\;& \frac{1}{\pi^2}\int_0^{+\infty}dz\int_0^{+\infty}\frac{f(x-t,y-z)-2f(x-t,y)+f(x-t,y+z)}{z^2t^2}dt+\\
&-\frac{2}{\pi^2}\int_0^{+\infty}dz\int_0^{+\infty}\frac{f(x,y-z)-2f(x,y)+f(x,y+z)}{z^2t^2}dt+\\
&+\frac{2}{\pi^2}\int_0^{+\infty}dz\int_0^{+\infty}\frac{f(x+t,y-z)-2f(x+t,y)+f(x+t,y+z)}{z^2t^2}dt.
\end{aligned}$$

It is easy to check that $\frac{\partial^2}{\partial|x|\partial|y|}f(x,y)$ produces the same result and thus the commutativity of the second-order fractional derivative holds. □

We consider now the last case of composition of Cauchy processes: it can be seen that the density of the process

$$C_1^a(|C_2(t)|) \qquad t>0,$$

where the external process is endowed with a position parameter $a\in\mathbb{R}$, is a solution to a non-homogeneous wave equation. Indeed, by suitably adapting the proof of theorem 4.1 in D'Ovidio and Orsingher (2009), it is easy to check that

$$p_{CC}(x,t)=\frac{2}{\pi^2}\int_0^{+\infty}\frac{s}{s^2+(x-a)^2}\frac{t}{t^2+s^2}ds$$

is a solution to

$$\frac{\partial^2 p}{\partial t^2} = \frac{\partial^2 p}{\partial x^2} - \frac{1}{\pi t (x-a)^2}, \qquad x \in \mathbb{R},\ t > 0. \tag{3.42}$$

In the d-dimensional case the iterated Cauchy process can be defined as follows

$$J^d_{CC}(t) = \begin{cases} C_1(|C(t)|) \\ \dots \\ C_d(|C(t)|) \end{cases}, \qquad t > 0 \tag{3.43}$$

where $C_1 \dots C_d$ and C are mutually independent, standard Cauchy processes.

Theorem 3.6 *The density of* $J^d_{CC}(t), t > 0$*, which can be expressed as*

$$p^d_{CC}(x_1, \dots, x_d, t) = \frac{2}{\pi^{d+1}} \int_0^{+\infty} \prod_{j=1}^{d} \frac{s}{s^2 + x_j^2} \frac{t}{t^2 + s^2} ds \tag{3.44}$$

and satisfies, for $d > 1$*, the following fractional equation*

$$\frac{\partial^2 p}{\partial t^2} = \sum_{k=1}^{d} \frac{\partial^2 p}{\partial x_k^2} - 2 \sum_{k=2}^{d} \sum_{j=1}^{k-1} \frac{\partial^2 p}{\partial |x_k| \partial |x_j|}, \qquad x_j \in \mathbb{R},\ j = 1, \dots, d,\ t > 0, \tag{3.45}$$

with initial condition $p(x_1, \dots, x_d, 0) = \prod_{j=1}^{d} \delta(x_j)$.

Proof The second-time derivative of (3.44) can be evaluated by adapting the proof of the previous theorem, as follows

$$\begin{aligned}
& \frac{\partial^2}{\partial t^2} p^d_{CC}(x_1, \dots, x_d, t) \\
= \ & \frac{2}{\pi^{d+1}} \int_0^{+\infty} \prod_{j=1}^{d} \frac{s}{s^2 + x_j^2} \frac{\partial^2}{\partial t^2} \left(\frac{t}{t^2 + s^2} \right) ds \\
= \ & -\frac{2}{\pi^{d+1}} \int_0^{+\infty} \prod_{j=1}^{d} \frac{s}{s^2 + x_j^2} \frac{\partial^2}{\partial s^2} \left(\frac{t}{t^2 + s^2} \right) ds \\
= \ & \frac{2}{\pi^{d+1}} \frac{t}{t^2 + s^2} \frac{\partial}{\partial s} \left(\prod_{j=1}^{d} \frac{s}{s^2 + x_j^2} \right) \Bigg|_0^{+\infty} + \\
& -\frac{2}{\pi^{d+1}} \int_0^{+\infty} \frac{t}{t^2 + s^2} \frac{\partial^2}{\partial s^2} \left(\prod_{j=1}^{d} \frac{s}{s^2 + x_j^2} \right) ds \\
= \ & [\text{by } (3.39)] \\
= \ & \sum_{k=1}^{d} \frac{\partial^2}{\partial x_k^2} p^d_{CC}(x_1, \dots, x_d, t) - 2 \sum_{k=2}^{d} \sum_{j=1}^{k-1} \frac{\partial^2}{\partial |x_k| \partial |x_j|} p^d_{CC}(x_1, \dots, x_d, t).
\end{aligned}$$

□

Remark 3.5 The density of $J_{CC}^d(t), t>0$ can be expressed in the following alternative form

$$p_{CC}^d(x_1,...,x_d,t)=\prod_{j=1}^d \frac{2t}{\pi^2\left(t^2+x_j^2\right)}\ln\frac{t}{|x_j|}, \tag{3.46}$$

as can be inferred from the calculations leading to theorem 4.1 of D'Ovidio and Orsingher (2009).

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